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Relativistic adiabatic approximation and geometric phase

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Abstract. A relativistic analogue of the quantum adiabatic approximation is developed for Klein–Gordon fields minimally coupled to electromagnetism, gravity and an arbitrary scalar potential. The corresponding adiabatic dynamical and geometrical phases are calculated. The method introduced in this paper avoids the use of an inner product on the space of solutions of the Klein–Gordon equation. Its practical advantages are demonstrated in the analysis of the relativistic Landau level problem and the rotating cosmic string.

1. Introduction

Quantum adiabatic approximation [1–3] is almost as old as quantum mechanics itself, yet it has not lost its importance as one of the few available tools for investigating the solution of the Schrödinger equation for explicitly time-dependent Hamiltonians. There are numerous publications on the subject of the quantum adiabatic approximation and its applications‡. One of the most remarkable of these is Berry's pioneering article on the adiabatic geometrical phase [5].

Soon after the publication of the early results on the Abelian [5] and non-Abelian [6] geometric phases, Aharonov and Anandan [7] showed that within the framework of non-relativistic quantum mechanics, one could introduce a geometric phase factor for arbitrary, not necessarily adiabatic, cyclic evolutions. This was followed by an important observation by Garrison and Chiao [8] who generalized the results of Aharonov and Anandan to arbitrary classical field theories. This required the presence of a gauge symmetry which provided a conserved charge. The latter was then used to define an inner product on the space of the classical fields. Alternatively, one could require the existence of an inner product directly [9]. A common assumption of [8, 9] was that the field equations involved only the first time derivative of the fields.

Clearly the simplest classical relativistic field is a Klein–Gordon field in the ordinary Minkowski spacetime. The phenomenon of the geometric phase for charged Klein–Gordon fields minimally coupled to a time-dependent electromagnetic field has been studied by Anandan and Mazur [10]. The main strategy of [10] is to decompose the vector space of the fields into three subspaces which are spanned respectively by the positive, zero, and negative frequency (energy) solutions, and to note that on the positive and negative frequency subspaces, where the Klein–Gordon inner product is positive, respectively, negative definite, the Klein–Gordon equation may be written as a pair of equations which are linear in the time-derivative of the field. More recently, a similar approach has been pursued to study the

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‡ A recent discussion and a list of important references are given in [4].

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dynamics of Klein–Gordon fields in a periodic Friedmann–Robertson–Walker background by Droz-Vincent [11].

One of the motivations for the study of the geometric part of the phase of a scalar field is the problem of time in quantum cosmology. The first developments in this direction, to the best of my knowledge, go back to the work of Brout and Venturi [12]. This was inspired by the earlier results of Banks [13] and Brout [14] on the use of the Born–Oppenheimer approximation in the semiclassical treatment of the Wheeler–DeWitt equation, and the application of Berry's phase in improving the Born–Oppenheimer approximation in molecular physics [15]. Subsequent work which followed essentially the same idea is that of Venturi [16, 17], Casadio and Venturi [18] and Datta [19]. There are also the contributions of Cai and Papini [20] which are based on the proper time or four-space formulation of relativistic quantum mechanics.

More recently Corichi and Pierri [21] considered Klein–Gordon fields in a class of stationary spacetimes and in particular investigated the induced topological Aharonov–Bohm type phases due to a rotating cosmic string. The analogy between the topological phase due to a rotating cosmic string and the Aharonov–Bohm phase had previously been pointed out by de Sousa Gerbert and Jackiw [22].

Although the original approach of Garrison and Chiao [8] does not require the evolution of the field to be adiabatic, as seen from the example studied by Anandan and Mazur [10], one cannot usually compute the geometric phase analytically without assuming the adiabaticity of the evolution. This suggests a systematic study of a possible relativistic generalization of the quantum adiabatic approximation. The main purpose of this article is to develop such a generalization for a charged Klein–Gordon field Φ in an arbitrary globally hyperbolic spacetime (M, g) which is minimally coupled to an electromagnetic potential A, as well as an arbitrary scalar potential V. The latter may, for instance, be identified with the appropriate multiple of the Ricci scalar curvature which renders the theory conformally invariant. The problem of the investigation of the dynamics of such a field theory has a long history in the context of developing quantum field theories in a curved background spacetime [23–25]. Here, I shall not be concerned with subtleties associated with the full second quantized theory. Instead, the Klein–Gordon field will be viewed and treated as a classical (first quantized) field.

In section 2, a two-component formulation of the field equation is described. This allows for a simple generalization of Berry's original approach [5] to the relativistic case and yields the relativistic analogues of the adiabatic approximation and the adiabatic dynamical and geometric phases. These are discussed in sections 3–5. It is shown that a direct generalization of the methods of non-relativistic quantum mechanics leads to an adiabaticity condition which unlike its non-relativistic counterpart also limits the rate of change of the energy eigenvalues. The corresponding approximation is, therefore, named the *ultra-adiabatic approximation*. Relaxing the condition on the energy eigenvalues and enforcing only the analogue of the non-relativistic adiabaticity condition, one obtains a more general notion of the relativistic adiabatic approximation. It turns out that the latter leads to the same expression for the geometric phase, but modifies the expression for the dynamical phase. In sections 6 and 7, the results of sections 2–5 are employed in the investigation of the geometric phases due to a rotating magnetic field in Minkowski space and a rotating cosmic string, respectively. Section 8 summarizes the conclusions.

2. Two-component formulation of the Klein–Gordon equation

Consider a complex scalar field Φ defined on a globally hyperbolic spacetime $(M, \mathrm{g}) = (\mathbb{R} \times \Sigma, \mathrm{g})$ satisfying

$$[g^{\mu\nu}(\nabla_\mu + \mathrm{i}eA_\mu)(\nabla_\nu + \mathrm{i}eA_\nu) + V - \mu^2]\Phi = 0 \tag{1}$$

where $g^{\mu\nu}$ are components of the inverse of the metric g, ∇_μ is the covariant derivative along $\partial/\partial x^\mu$ defined by the Levi-Civita connection, A_μ are components of the electromagnetic potential, V is an arbitrary scalar potential, e is the electric charge, and μ is the mass. Throughout this article the signature of the metric g is chosen to be $(-,+,+,+)$ and letters from the Greek alphabet are associated with a local coordinate basis of the tangent spaces (bundle) of the spacetime manifold. The letters from the Latin alphabet label the corresponding spatial components. They take values of 1, 2 and 3.

Denoting a time derivative by a dot, one can express equation (1) in the form

$$\ddot{\Phi} + \hat{D}_1\dot{\Phi} + \hat{D}_2\Phi = 0 \tag{2}$$

where

$$\hat{D}_1 := \frac{2}{g^{00}}[g^{0i}\partial_i + \mathrm{i}eg^{0\mu}A_\mu - \tfrac{1}{2}\, g^{\mu\nu}\Gamma^0_{\mu\nu}] \tag{3}$$

$$\hat{D}_2 := \frac{2}{g^{00}}[\tfrac{1}{2}\, g^{ij}\partial_i\partial_j + (\mathrm{i}eg^{\mu i}A_\mu - \tfrac{1}{2}g^{\mu\nu}\Gamma^i_{\mu\nu})\partial_i + \tfrac{1}{2}\, g^{\mu\nu}(\mathrm{i}e\nabla_\mu A_\nu - e^2A_\mu A_\nu) + \tfrac{1}{2}(V - \mu^2)]. \tag{4}$$

A two-component representation of the field equation (2) is

$$\mathrm{i}\dot{\Psi}^{(q)} = \hat{H}^{(q)}\Psi^{(q)} \tag{5}$$

where

$$\Psi^{(q)} := \begin{pmatrix} u^{(q)} \\ v^{(q)} \end{pmatrix} \tag{6}$$

$$u^{(q)} := \frac{1}{\sqrt{2}}(\Phi + q\dot{\Phi}) \qquad v^{(q)} := \frac{1}{\sqrt{2}}(\Phi - q\dot{\Phi}) \tag{7}$$

$$\hat{H}^{(q)} := \frac{\mathrm{i}}{2}\begin{pmatrix} \frac{\dot{q}}{q} + \frac{1}{q} - \hat{D}_1 - q\hat{D}_2 & -\frac{\dot{q}}{q} - \frac{1}{q} + \hat{D}_1 - q\hat{D}_2 \\ -\frac{\dot{q}}{q} + \frac{1}{q} + \hat{D}_1 + q\hat{D}_2 & \frac{\dot{q}}{q} - \frac{1}{q} - \hat{D}_1 + q\hat{D}_2 \end{pmatrix} \tag{8}$$

and q is an arbitrary, possibly time-dependent, non-zero complex parameter. The set $\mathbb{C} - \{0\}$ of q's defines a group of transformations

$$\Psi^{(q)} \to \Psi^{(q')} =: g(q', q)\Psi^{(q)} \tag{9}$$

which is isomorphic to $GL(1, \mathbb{C})$. The group elements are given by

$$g(q', q) = g(\gamma) = \begin{pmatrix} \frac{1+\gamma}{2} & \frac{1-\gamma}{2} \\ \frac{1-\gamma}{2} & \frac{1+\gamma}{2} \end{pmatrix}$$

where $\gamma := q'/q$. Under the transformation (9), the Hamiltonian transforms according to

$$\hat{H}^{(q)} \to \hat{H}^{(q')} = g(\gamma)\hat{H}^{(q)}g^{-1}(\gamma) + \mathrm{i}\dot{g}(\gamma)g^{-1}(\gamma)$$

and the Schrödinger equation (5) preserves its form. The underlying $GL(1, \mathbb{C})$ symmetry which characterizes the arbitrariness of q does not have any physical significance. It is, however, useful for computational purposes as shown in [26].

The advantage of the two-component form of the field equation is that it enables one to proceed in a manner analogous with the well known non-relativistic quantum mechanical

case. Indeed equation (5) with a fixed choice of q is a Schrödinger equation associated with an explicitly time-dependent Hamiltonian $\hat{H}^{(q)}$. The two-component fields $\Psi^{(q)}$ belong to the vector space $\mathcal{H}_t \oplus \mathcal{H}_t$ where $\mathcal{H}_t$ is the Hilbert space completion (with respect to an appropriate inner product) of compactly supported complex-valued functions on the spatial hypersurface Σ_t associated with a specific Arnowitt–Deser–Misner (ADM) decomposition of the spacetime [27].

Usually in the two-component approach to the Klein–Gordon field theory in Minkowski spacetime, one chooses an inner product on $\mathcal{H}_t \oplus \mathcal{H}_t$ in such a way as to make the corresponding Hamiltonian self-adjoint [28, 29]. A Hermitian inner product $(\ ,\ )$ on $\mathcal{H}_t \oplus \mathcal{H}_t$ may be defined by a Hermitian inner product $\langle\ |\ \rangle$ on $\mathcal{H}_t$ and a possibly time-dependent complex Hermitian 2×2 matrix $h = (h_{rs})$:

$$(\Psi_1, \Psi_2) := (\langle u_1|, \langle v_1|) \begin{pmatrix} h_{11} & h_{12} \\ h_{12}^* & h_{22} \end{pmatrix} \begin{pmatrix} |u_2\rangle \\ |v_2\rangle \end{pmatrix} \tag{10}$$

where u_r and v_r are components of Ψ_r, and h_{11} and h_{22} are real. The usual choice for h, in the Minkowski case, is [28, 29]

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This choice leads to

$$(\Psi_1, \Psi_2) = \langle u_1|u_2\rangle - \langle v_1|v_2\rangle. \tag{11}$$

It is not difficult to check that in the general case this choice does not guarantee the self-adjointness of the Hamiltonian unless some severe conditions are imposed on q and the operators $\hat{D}_1$ and $\hat{D}_2$, namely, that q must be imaginary, $\hat{D}_2$ must be self-adjoint with respect to the inner product $\langle\ |\ \rangle$ on $\mathcal{H}_t$, and $\hat{D}_1 = \dot{q}/q$. The latter condition is especially restrictive as q can only depend on time and being a free (non-dynamical) parameter, may be set to a constant in which case $\hat{D}_1$ must vanish. In general, these conditions are not fulfilled. Nevertheless, the inner product (11) has an appealing property which is described next.

Consider the eigenvalue problem for $H^{(q)}$. Denoting the eigenvalues and eigenvectors by $E_n^{(q)}$ and $\Psi_n^{(q)}$, i.e.

$$H^{(q)}\Psi_n^{(q)} = E_n^{(q)}\Psi_n^{(q)} \tag{12}$$

expressing $\Psi_n^{(q)}$ in two-component form, and using equation (8), one can easily show that up to an undetermined scalar multiple, $\Psi_n^{(q)}$ has the following form:

$$\Psi_n^{(q)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - \mathrm{i}q\,E_n^{(q)} \\ 1 + \mathrm{i}q\,E_n^{(q)} \end{pmatrix} \Phi_n^{(q)} \tag{13}$$

where $\Phi_n^{(q)} \in \mathcal{H}_t$ satisfies

$$\left[\hat{D}_2 - \mathrm{i}E_n^{(q)}\left(\hat{D}_1 - \frac{\dot{q}}{q}\right) - (E_n^{(q)})^2\right]\Phi_n^{(q)} = 0. \tag{14}$$

This equation may be viewed as a 'generalized' eigenvalue equation† in $\mathcal{H}_t$. It defines both the vectors $\Phi_n^{(q)}$ and the complex numbers $E_n^{(q)}$. It reduces to the ordinary eigenvalue equation for $\hat{D}_2$, if $\hat{D}_1 = \dot{q}/q$. Note that this is also one of the conditions for self-adjointness of the Hamiltonian, with the choice of (11) for the inner product. Furthermore,

† Note that this terminology has nothing to do with the concept of generalized eigenvectors of spectral analysis.

if this condition is satisfied, then equation (14) determines $E_n^{(q)}$ up to a sign, i.e. eigenvalues come in pairs of opposite sign.

If q is chosen to be time-independent, then (14) does not carry any information about q and therefore $\Phi_n^{(q)}$ and $E_n^{(q)}$ are independent of the choice of q†. Hence, one can drop the labels (q) on the right-hand side of equation (13). In this case, equation (14) becomes

$$[\hat{D}_2 - \mathrm{i}E_n\hat{D}_1 - E_n^2]\Phi_n = 0. \tag{15}$$

Now let us use the inner product (11) to compute the inner product of two eigenvectors of the Hamiltonian. Performing the algebra, one finds

$$(\Psi_m^{(q)}, \Psi_n^{(q)}) = \mathrm{i}(q^*E_m^* - qE_n)\langle\Phi_m|\Phi_n\rangle. \tag{16}$$

Therefore if q is a positive imaginary number, i.e. $q = \mathrm{i}|q|$, then

$$(\Psi_m^{(q)}, \Psi_n^{(q)}) = |q|(E_m^* + E_n)\langle\Phi_m|\Phi_n\rangle. \tag{17}$$

Hence the eigenvectors $\Psi_n^{(q)}$ and $\Psi_m^{(q)}$ with $E_m = -E_n^*$ (if they exist) are orthogonal regardless of the value of $\langle\Phi_m|\Phi_n\rangle$. Furthermore, one has $(\Psi_n^{(q)}, \Psi_n^{(q)}) = 2|q|\mathrm{Re}(E_n)\langle\Phi_n|\Phi_n\rangle$, i.e. the norm of an energy eigenvector has the same sign as the real part of the corresponding eigenvalue. It vanishes for the zero and imaginary energy eigenvalues. Note that here I am assuming that the inner product $\langle\ |\ \rangle$ on $\mathcal{H}_t$ is non-negative. In fact $\mathcal{H}_t$ is to be identified with the separable Hilbert space $L^2(\Sigma_t)$ of square-integrable functions on Σ_t where the integration is defined by the measure $[\det({}^{(3)}\mathrm{g})]^{1/2}$ given by the Riemannian three-metric ${}^{(3)}\mathrm{g}$. The latter is induced by the four-metric g.

Another interesting property of the inner product (11) is the fact that for imaginary q it yields the familiar Klein–Gordon inner product, $\langle\ ,\ \rangle_{\mathrm{KG}}$. This is easily seen by substituting (7) in (11), which leads to

$$(\Psi_1, \Psi_2) = q\langle\Phi_1|\dot{\Phi}_2\rangle + q^*\langle\dot{\Phi}_1|\Phi_2\rangle = q[\langle\Phi_1|\dot{\Phi}_2\rangle - \langle\dot{\Phi}_1|\Phi_2\rangle] =: q\langle\Phi_1, \Phi_2\rangle_{\mathrm{KG}}. \tag{18}$$

It is also useful to recall that the space $\mathcal{H}_t \oplus \mathcal{H}_t$ is nothing but the space of the possible initial conditions $[\Phi(t, x^i), \dot{\Phi}(t, x^i)]$ with initial time being t and $(x^i) \in \Sigma_t$. In view of the well-posedness of the dynamical equation [30], this (vector) space is isomorphic to the space of solutions of the field equation (1). Hence a two-component decomposition may be viewed as a splitting of the space of solutions of the field equations. In view of the freedom of choice of the parameter q, this splitting is clearly not unique.

3. Cyclic states and quantal phases

By definition a state (an element of the projective Hilbert space) of a quantum mechanical system, whose dynamics is governed by the Schrödinger equation

$$\mathrm{i}\dot{\psi}(t) = \hat{H}(t)\psi(t) \tag{19}$$

is said to be cyclic with a period τ, if it is an eigenstate of the time-evolution operator $\hat{U}(\tau) := \mathcal{T}\exp[-\mathrm{i}\int_0^\tau \hat{H}(t)\,\mathrm{d}t]$. Here $\mathcal{T}$ is the time-ordering operator. An associated initial state vector $\psi(0)$ then satisfies

$$\psi(\tau) = \hat{U}(\tau)\psi(0) = \mathrm{e}^{\mathrm{i}\alpha(\tau)}\psi(0) \tag{20}$$

where $\alpha(\tau) \in \mathbb{C}$. If the Hamiltonian is self-adjoint, then $\alpha(\tau) \in \mathbb{R}$ and consequently $\psi(\tau)$ and $\psi(0)$ differ in phase. In general $\alpha(\tau)$ may be expressed as the sum of a dynamical and

† This can also be seen by noting that under the transformation $\Psi_n^{(q)} \to \tilde{\Psi}_n^{(q)} = g(q', q)\Psi_n^{(q)}$, the eigenvectors preserve their form (13) and that $\tilde{\Psi}_n^{(q)}$ is an eigenvector of $\hat{H}^{(q')}$ with the same eigenvalue $E_n^{(q)}$.

a geometrical part [7]. This decomposition uses the inner product structure of the Hilbert space.

The situation is rather more transparent, if the time-dependence of the Hamiltonian is adiabatic. In this case, one can follow Berry's approach [5] of employing the adiabatic theorem of quantum mechanics. According to the adiabatic theorem [31],

> *'if the initial state is an eigenstate of the initial Hamiltonian $\hat{H}(0)$, then in time $t > 0$ the evolving state remains an eigenstate of the Hamiltonian $\hat{H}(t)$.'* (21)

More precisely, assume that the time-dependence of the Hamiltonian is realized through its dependence on a set of parameters $R = (R^1, \ldots, R^n)$ and a smooth curve $C : [0, \tau] \to \mathcal{M}$, where R is viewed as coordinates of a parameter space $\mathcal{M}$, i.e. $\hat{H}(t) := H[R(t)]$, and $R(t) = (R^1(t), \ldots, R^n(t)) := C(t)$. Furthermore, let $\psi_n[R]$ denote eigenvectors of $\hat{H}[R]$ with eigenvalue $E_n[R]$, i.e.

$$\hat{H}[R]\psi_n[R] = E_n[R]\psi_n[R] \tag{22}$$

and suppose that

(a) ψ_n and E_n are smooth functions of R;

(b) for all n the degree of degeneracy of E_n is independent of R; and

(c) no level crossings occur during the evolution of the system.

Then the statement of the adiabatic theorem may be summarized by the relation

$$\psi(t) := \hat{U}(t)\psi_n(0) \approx \mathrm{e}^{\mathrm{i}\alpha_n(t)}\psi_n(t) \tag{23}$$

where $\psi_n(t) := \psi_n[R(t)]$. If $E_n(t) := E_n[R(t)]$ is $\mathcal{N}$-fold degenerate, then ψ_n belongs to the $\mathcal{N}$-dimensional degeneracy subspace $\mathcal{H}_n$ and α_n is an $\mathcal{N} \times \mathcal{N}$ matrix-valued function of time. The approximation sign $\approx$ in (23) is used to emphasize that this relation is only valid if the adiabatic approximation is justified.

Assuming the validity of the adiabatic approximation ($\approx \to =$) and substituting (23) in the Schrödinger equation (19), one has [6]

$$\mathrm{e}^{\mathrm{i}\alpha_n(t)} = \exp\left[-\mathrm{i}\int_0^t E_n(t')\,\mathrm{d}t'\right]\mathcal{P}\exp\left[\mathrm{i}\int_{C(0)}^{C(t)}\mathcal{A}_n\right] \tag{24}$$

$$\mathcal{A}_n^{IJ}[R] := \frac{\mathrm{i}\langle\psi_n^I[R],\, \frac{\partial}{\partial R^a}\psi_n^J[R]\rangle}{\langle\psi_n^I[R],\, \psi_n^I[R]\rangle}\,\mathrm{d}R^a = \frac{\mathrm{i}\langle\psi_n^I[R],\, \mathrm{d}\psi_n^J[R]\rangle}{\langle\psi_n^I[R],\, \psi_n^I[R]\rangle} \tag{25}$$

where $\mathcal{P}$ is the path-ordering operator, $\psi_n^I[R]$ form a complete orthogonal basis of the degeneracy subspace $\mathcal{H}_n$, and $\langle\ ,\ \rangle$ is the inner product. If the Hamiltonian is periodic, i.e. C is a closed curve, then according to (23), $\psi_n[R(0)] = \psi_n[R(T)]$ is a cyclic state vector. In this case the first and the second (path-ordered) exponential in (24), with $t = T$, are called the *dynamical* and the *geometrical* parts of the total adiabatic matrix-valued phase $\exp[\mathrm{i}\alpha_n(T)]$, respectively [6]. The qualification 'geometrical' is best justified by identifying the geometric part of the phase by the holonomy of a principal spectral bundle over the parameter space $\mathcal{M}$ or alternatively the universal classifying bundle over the projective Hilbert space, [32, 7, 33].

The situation for a non-self-adjoint Hamiltonian is rather more complicated. The dynamical and the geometrical phase can still be defined in terms of the projective Hilbert space [34]. However, in general the eigenvectors of the Hamiltonian are not orthogonal†. This renders the proof of the adiabatic theorem [3, 31] invalid. One can still adopt (23)

† Note that the eigenvectors within a single degeneracy subspace can always be orthonormalized. However, the eigenvectors corresponding to distinct eigenvalues in general overlap.

as an ansatz which may or may not be valid for specific evolutions. The condition of the validity of this ansatz, which allows one to pursue the same strategy in defining the adiabatic geometric phase, is

$$\langle\psi_m, \psi_n\rangle\langle\psi_n, \dot{\psi}_n\rangle = \langle\psi_m, \dot{\psi}_n\rangle\langle\psi_n, \psi_n\rangle \tag{26}$$

where ψ_m and ψ_n are any pair of distinct eigenvectors of the Hamiltonian. This condition is obtained by substituting (23) in the Schrödinger equation (19) and taking the inner product of both sides of the resulting equation with ψ_m. Equation (26) is trivially satisfied for the case of a self-adjoint Hamiltonian. In this case, the left-hand side vanishes identically and $\langle\psi_m, \dot{\psi}_n\rangle$ for $m \neq n$, vanishes approximately by virtue of the adiabatic approximation [4]. Hence, if one adopts the statement (21) as the definition of the adiabatic approximation also for the non-self-adjoint Hamiltonians, then instead of the conventional adiabaticity condition

$$\langle\psi_m, \dot{\psi}_n\rangle \approx 0 \qquad \text{for } m \neq n \tag{27}$$

one has the more general adiabaticity condition

$$\langle\psi_m, \psi_n\rangle\langle\psi_n, \dot{\psi}_n\rangle - \langle\psi_m, \dot{\psi}_n\rangle\langle\psi_n, \psi_n\rangle \approx 0. \tag{28}$$

Before pursuing the derivation of the expression for the geometric phase, I must emphasize that a general cyclic two-component state vector is clearly cyclic in both its components. Identifying the corresponding function space $\mathcal{H}_t \oplus \mathcal{H}_t$ (note that the t-dependence is only relevant to the inner product structure and the vector space structure is independent of t) with the space of all possible initial data, a cyclic two-component state vector $\Psi^{(q)}(0)$ which by definition satisfies $\Psi^{(q)}(\tau) = \exp[\mathrm{i}\alpha(\tau)]\Psi^{(q)}(0)$, is associated with a 'cyclic' solution of the Klein–Gordon equation (1) whose velocity is also cyclic with the same (possibly non-unimodular) 'phase' and period, i.e.

$$\Phi(\tau, x^i) = \mathrm{e}^{\mathrm{i}\alpha(\tau)}\Phi(0, x^i) \qquad \dot{\Phi}(\tau, x^i) = \mathrm{e}^{\mathrm{i}\alpha(\tau)}\dot{\Phi}(0, x^i). \tag{29}$$

This is in contrast with the usual definition of a cyclic evolution for classical fields [8, 9]. It seems more reasonable to ascribe the term 'cyclic' to a repetition, up to a scalar multiple, of both the initial conditions, i.e.

$$\Phi(\tau, x^i) = \exp[\mathrm{i}\alpha(\tau)]\Phi(0, x^i) \qquad \dot{\Phi}(\tau, x^i) = \exp[\mathrm{i}\beta(\tau)]\dot{\Phi}(0, x^i) \tag{30}$$

where α and β may or may not be equal. In this article I shall use the term *cyclic* in this sense. If the stronger condition (29) is satisfied, i.e. if $\beta = \alpha$, then the evolution will be called *ultracyclic*.

4. Relativistic ultra-adiabatic approximation

Consider the two-component formulation of the Klein–Gordon equation. Suppose for simplicity that $E_n^{(q)}$ of equation (12) is independent of q, i.e. $E_n^{(q)} = E_n$ and that it is non-degenerate. Then, a direct generalization of the concept of adiabatic evolution in non-relativistic quantum mechanics (21) prompts one to use the ansatz

$$\Psi^{(q)}(t) \approx \mathrm{e}^{\mathrm{i}\alpha_n(t)}\Psi_n^{(q)}[R(t)] \tag{31}$$

as the defining relation for the relativistic adiabatic evolution. One can show, however, that this ansatz leads to a rather restrictive notion of adiabatic approximation. I shall refer to this approximation as the *ultra-adiabatic approximation.* More precisely, I shall adopt the following definition.

Definition 1. A two-component state vector $\Psi^{(q)}(t)$ is said to undergo an *exact ultra-adiabatic evolution* if and only if

$$\Psi^{(q)}(t) = \mathrm{e}^{\mathrm{i}\alpha_n(t)}\Psi_n^{(q)}[R(t)] \tag{32}$$

for some n and α_n.

Note that definition 1 also provides a definition for *ultra-adiabatic approximation* by replacing equation (32) by (31).

In order to derive the conditions under which the ultra-adiabatic approximation is valid, one must substitute equations (32), (12), and (13) in the Schrödinger equation (5). This yields

$$[-\dot\alpha_n(1-\mathrm{i}qE_n) + q\dot E_n - E_n(1-\mathrm{i}qE_n)]\Phi_n + \mathrm{i}(1-\mathrm{i}qE_n)\dot\Phi_n = 0 \tag{33}$$

$$[-\dot\alpha_n(1+\mathrm{i}qE_n) - q\dot E_n - E_n(1+\mathrm{i}qE_n)]\Phi_n + \mathrm{i}(1+\mathrm{i}qE_n)\dot\Phi_n = 0. \tag{34}$$

Adding both sides of these equations and simplifying the result, one has

$$(\dot\alpha_n + E_n)\Phi_n - \mathrm{i}\dot\Phi_n = 0. \tag{35}$$

This equation leads directly to the expression for the total phase (24) with the Berry connection one-form given by

$$\mathcal{A}_n = \frac{\mathrm{i}\langle\Phi_n|\frac{\partial}{\partial R^a}\Phi_n\rangle}{\langle\Phi_n|\Phi_n\rangle}\,\mathrm{d}R^a = \frac{\mathrm{i}\langle\Phi_n|\mathrm{d}\Phi_n\rangle}{\langle\Phi_n|\Phi_n\rangle}. \tag{36}$$

Here R denotes the parameters of the system, i.e. the metric g, the electromagnetic potential A and the scalar potential V. Moreover I have used the identity $\dot\Phi_n\,\mathrm{d}t = (\partial\Phi_n/\partial R^a)\,\mathrm{d}R^a = \mathrm{d}\Phi_n$.

Furthermore, subtracting equation (34) from (33) and using equation (35) to simplify the resulting expression, one finds

$$\dot E_n = 0. \tag{37}$$

This condition which is a direct consequence of definition 1 does not have a counterpart in ordinary non-relativistic quantum mechanics. Its roots may be sought in the fundamental difference between ordinary (one-component) Schrödinger and Klein–Gordon equations. One might argue that the condition (37) and consequently the concept of the ultra-adiabatic evolution are too restrictive. Indeed it is possible to relax this condition by adopting a more general definition of adiabatic evolution. For the moment, however, I shall continue with a further analysis of the ultra-adiabatic evolutions.

Because equation (35) is identical with the one obtained in the non-relativistic case, in addition to condition (37) one also has the analogue of equation (26). If Φ_n turn out to be orthogonal, the latter reduces to

$$\langle\Phi_m|\dot\Phi_n\rangle = 0 \qquad \forall m \neq n \tag{38}$$

which is the well known condition for the exactness of the adiabatic approximation in non-relativistic quantum mechanics [4]. Hence, for the cases where Φ_n are orthogonal, the ultra-adiabatic approximation is exact if and only if equations (37) and (38) are satisfied.

Similarly to the non-relativistic case, the condition of the exactness of ultra-adiabatic approximation is highly restrictive. In fact, the ultra-adiabatic approximation is exact, if and only if the evolving state is stationary [35]. More interesting are cases where the ultra-adiabatic approximation is valid only approximately, i.e. cases where instead of (32), (31)

holds. In this case, equations (33), (34), and conditions (37) and (38) are required to be satisfied approximately, namely

$$\dot{E}_n \approx 0 \tag{39}$$

$$\langle \Phi_m | \dot{\Phi}_n \rangle \approx 0 \qquad \forall m \neq n. \tag{40}$$

The precise meaning of the $\approx$ in these equations will be clarified momentarily.

In the above discussion, the condition of time-independence of q does not play any significant role in the derivation of equations (35) and (36). In fact, allowing q to be time-dependent only changes the term $q\dot{E}_n$ in equations (33) and (34) to $\mathrm{d}(qE_n^{(q)})/\mathrm{d}t$. Therefore, upon adding the resulting equations one still obtains equation (35). The only consequences of using a time-dependent q are the emergence of q-dependent E_n and Φ_n and the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}(qE_n^{(q)}) \approx 0 \tag{41}$$

which generalizes (39).

There is a particular case in which q may be time-dependent but E_n and Φ_n are still independent of the choice of q. This is the case, where the operator $\hat{D}_1$ of (3) is zeroth order and it only involves time-dependent functions. In this case one can choose q in such a way as to satisfy $\hat{D}_1 = \dot{q}/q$. This condition reduces equation (14) to the eigenvalue equation for $\hat{D}_2$, with eigenvalues E_n^2 and eigenvectors Φ_n. Hence, E_n and Φ_n are still q-independent. In [26], it is shown how this apparently very special case may be realized and used in the study of spatially homogeneous (Bianchi) cosmological models.

5. Relativistic adiabatic approximation

The appearance of the decomposition parameter q in (41) and the fact that this condition has no non-relativistic analogue suggests that perhaps the notion of ultra-adiabatic approximation is too limited. In order to obtain a more appealing concept of adiabatic approximation, one must consider a more general ansatz than (31).

Consider the general solutions Ψ of the two-component Schrödinger equation (5) of the form

$$\Psi^{(q)} = \sum_n \mathrm{e}^{\mathrm{i}\alpha_n} \Psi_n^{(q)} \tag{42}$$

where $\alpha_n \in \mathbb{C}$ and $\Psi_n^{(q)}$ are the eigenvectors of the two-component Hamiltonian (8). Substituting equation (42) in the Schrödinger equation (5) and making use of equations (12) and (13), one has

$$\sum_n \mathrm{e}^{\mathrm{i}\alpha_n} \left\{ \left[(E_n^{(q)} + \dot{\alpha}_n)(1 + \mathrm{i}qE_n^{(q)}) + \frac{\mathrm{d}}{\mathrm{d}t}(qE_n^{(q)}) \right] \Phi_n^{(q)} - \mathrm{i}(1 - \mathrm{i}qE_n^{(q)})\dot{\Phi}_n^{(q)} \right\} = 0 \tag{43}$$

$$\sum_n \mathrm{e}^{\mathrm{i}\alpha_n} \left\{ \left[(E_n^{(q)} + \dot{\alpha}_n)(1 - \mathrm{i}qE_n^{(q)}) - \frac{\mathrm{d}}{\mathrm{d}t}(qE_n^{(q)}) \right] \Phi_n^{(q)} - \mathrm{i}(1 + \mathrm{i}qE_n^{(q)})\dot{\Phi}_n^{(q)} \right\} = 0. \tag{44}$$

Adding and subtracting both sides of these equations and simplifying the result lead to

$$\sum_n \mathrm{e}^{\mathrm{i}\alpha_n} [(E_n^{(q)} + \dot{\alpha}_n)\Phi_n^{(q)} - \mathrm{i}\dot{\Phi}_n^{(q)}] = 0 \tag{45}$$

$$\sum_n \mathrm{e}^{\mathrm{i}\alpha_n} \left\{ \left[\frac{\mathrm{d}}{\mathrm{d}t}(qE_n^{(q)}) \right] \Phi_n^{(q)} + \mathrm{i}qE_n^{(q)} [(E_n^{(q)} + \dot{\alpha}_n)\Phi_n^{(q)} - \mathrm{i}\dot{\Phi}_n^{(q)}] \right\} = 0. \tag{46}$$

Next assume that $\hat{D}_2$ is a non-degenerate self-adjoint operator with a discrete spectrum and $\hat{D}_1 = \dot{q}/q$. Then, E_n and Φ_n do not depend on q and $\hat{D}_2\Phi_n = E_n^2\Phi_n$. Now, differentiate both sides of the latter equation with respect to time and take their inner product with Φ_m. Since in this case Φ_n are orthogonal, one has the well known identity [5]

$$\frac{\langle\Phi_m|\dot{\hat{D}}_2|\Phi_n\rangle}{E_n^2 - E_m^2} = \langle\Phi_m|\dot{\Phi}_n\rangle \qquad \text{for all } m \neq n \tag{47}$$

where Φ_n and Φ_m correspond to distinct eigenvalues of $\hat{D}_2$, i.e. $E_n^2 \neq E_m^2$. The quantum adiabatic approximation is valid if the left-hand side of this equation which involves the time-derivative of $\hat{D}_2$ can be neglected [4]. This statement provides the true meaning of the condition (40)

$$\langle\Phi_m|\dot{\Phi}_n\rangle = \frac{\langle\Phi_m|\dot{\hat{D}}_2|\Phi_n\rangle}{E_n^2 - E_m^2} \approx 0 \qquad \text{for all } m \neq n \tag{48}$$

for the case where the above assumptions are valid.

In the rest of this section I shall use the adiabaticity condition (40) to define the notion of *adiabatic evolution* in relativistic scalar quantum mechanics. Furthermore I shall assume that E_n are independent of q, $\hat{D}_2$ is a self-adjoint operator with a non-degenerate discrete spectrum, and that the energy eigenvalues come in opposite signs, i.e. $E_{\pm n} = \pm E_n$. This is the case if $\hat{D}_1 = \dot{q}/q$.

For convenience, I shall use the notation Ψ_{-n} for the two-component eigenvector corresponding to the eigenvalue $E_{-n} := -E_n$. Since for each pair $(-n, n)$ there is a single Φ_n, one can write equations (45) and (46) in the form

$$\sum_{n\geqslant 0}\{[E_n(\mathrm{e}^{\mathrm{i}\alpha_n} - \mathrm{e}^{\mathrm{i}\alpha_{-n}}) + (\dot{\alpha}_n\mathrm{e}^{\mathrm{i}\alpha_n} + \dot{\alpha}_{-n}\mathrm{e}^{\mathrm{i}\alpha_{-n}})]\Phi_n - \mathrm{i}(\mathrm{e}^{\mathrm{i}\alpha_n} + \mathrm{e}^{\mathrm{i}\alpha_{-n}})\dot{\Phi}_n\} = 0 \tag{49}$$

$$\sum_{n\geqslant 0}\left\{\left[(\mathrm{e}^{\mathrm{i}\alpha_n} - \mathrm{e}^{\mathrm{i}\alpha_{-n}})\frac{\mathrm{d}}{\mathrm{d}t}(qE_n) + \mathrm{i}qE_n^2(\mathrm{e}^{\mathrm{i}\alpha_n} + \mathrm{e}^{\mathrm{i}\alpha_{-n}}) + \mathrm{i}qE_n(\dot{\alpha}_n\mathrm{e}^{\mathrm{i}\alpha_n} - \dot{\alpha}_{-n}\mathrm{e}^{\mathrm{i}\alpha_{-n}})\right]\right.$$
$$\left. + qE_n(\mathrm{e}^{\mathrm{i}\alpha_n} - \mathrm{e}^{\mathrm{i}\alpha_{-n}})\dot{\Phi}_n\right\} = 0. \tag{50}$$

Enforcing condition (40), one can reduce (49) and (50) to

$$E_n(\mathrm{e}^{\mathrm{i}\alpha_n} - \mathrm{e}^{\mathrm{i}\alpha_{-n}}) + (\dot{\alpha}_n\mathrm{e}^{\mathrm{i}\alpha_n} + \dot{\alpha}_{-n}\mathrm{e}^{\mathrm{i}\alpha_{-n}}) - (\mathrm{e}^{\mathrm{i}\alpha_n} + \mathrm{e}^{\mathrm{i}\alpha_{-n}})a_n \approx 0 \tag{51}$$

$$(-\mathrm{i}f_n - a_n)(\mathrm{e}^{\mathrm{i}\alpha_n} - \mathrm{e}^{\mathrm{i}\alpha_{-n}}) + E_n(\mathrm{e}^{\mathrm{i}\alpha_n} + \mathrm{e}^{\mathrm{i}\alpha_{-n}}) + \dot{\alpha}_n\mathrm{e}^{\mathrm{i}\alpha_n} - \dot{\alpha}_{-n}\mathrm{e}^{\mathrm{i}\alpha_{-n}} \approx 0 \tag{52}$$

where $n \geqslant 0$ and

$$a_n := \frac{\mathrm{i}\langle\Phi_n|\dot{\Phi}_n\rangle}{\langle\Phi_n|\Phi_n\rangle} \qquad f_n := \frac{\frac{\mathrm{d}}{\mathrm{d}t}(qE_n)}{qE_n} = \frac{\mathrm{d}}{\mathrm{d}t}\ln(qE_n).$$

Adding and subtracting both sides of (51) and (52) and assuming that $\mathrm{e}^{\mathrm{i}\alpha_n}$ is not negligibly small, one finds

$$-\mathrm{i}f_n(1 - \mathrm{e}^{-\mathrm{i}(\alpha_n - \alpha_{-n})}) + 2(E_n + \dot{\alpha}_n - a_n) \approx 0 \tag{53}$$

$$-\mathrm{i}f_n(\mathrm{e}^{\mathrm{i}(\alpha_n - \alpha_{-n})} - 1) + 2(E_n - \dot{\alpha}_{-n} + a_n) \approx 0. \tag{54}$$

Next, define $\eta_n^- := \alpha_n - \alpha_{-n}$, add both sides of (53) and (54), and simplify the result. This leads to

$$\dot{\eta}_n^- + f_n\sin\eta_n^- + 2E_n \approx 0. \tag{55}$$

Introducing $\eta_n^+ := \alpha_n + \alpha_{-n}$ and using (55), one can then express (53) in the form

$$\dot{\eta}_n^+ - i f_n(1 - \cos\eta_n^-) - 2a_n \approx 0 \tag{56}$$

Hence in view of the definition $\eta_n^\pm := \alpha_n \pm \alpha_{-n}$, one has

$$\alpha_{\pm n}(t) = \tfrac{1}{2}[\eta_n^+(t) \pm \eta_n^-(t)] \approx \tfrac{1}{2}[\alpha_n(0) + \alpha_{-n}(0)] + \gamma_n(t) + \delta_{\pm n}(t) \qquad \forall n \geqslant 0 \tag{57}$$

$$\gamma_n(t) := \int_0^t a_n(t')\,\mathrm{d}t' = \int_{R(0)}^{R(t)} \mathcal{A}_n[R] \tag{58}$$

$$\delta_{\pm n}(t) := \mathrm{i}\xi(t) \pm \frac{\eta_n(t)}{2} \tag{59}$$

where I have used equation (36), η_n is the solution of

$$\dot{\eta}_n + f_n \sin\eta_n + 2E_n = 0 \qquad \text{with } \eta_n(0) = \alpha_n(0) - \alpha_{-n}(0) \tag{60}$$

and

$$\xi(t) := \tfrac{1}{2}\int_0^t f_n(t')[1 - \cos\eta_n(t')]\,\mathrm{d}t'. \tag{61}$$

As seen from (57)–(59), the part γ_n of $\alpha_{\pm n}$ which is independent of E_n has the same form as the geometric phase angle of the non-relativistic quantum mechanics. In contrast, the part $\delta_{\pm n}$ of $\alpha_{\pm n}$ which does depend on $E_{\pm n}$ and plays the role of the dynamical phase angle, has a different expression from its non-relativistic counterpart. For the case of an ultra-adiabatic evolution where condition (41) is satisfied, $f_n \approx 0$ and

$$\delta_{\pm n} = \mp\int_0^t E_n(t')\,\mathrm{d}t' \pm \tfrac{1}{2}[\alpha_n(0) - \alpha_{-n}(0)] = -\int_0^t E_{\pm n}(t')\,\mathrm{d}t' \pm \tfrac{1}{2}[\alpha_n(0) - \alpha_{-n}(0)].$$

Besides the unimportant constant term, this is identical with the expression for the non-relativistic adiabatic dynamical phase angle.

The above analysis shows that taking (40) as the defining condition for the *adiabatic approximation*, one obtains the same expression for the geometric phase as in the ultra-adiabatic case. This condition modifies the expression for the dynamical phase. In fact, the dynamical phase angle splits into a pair of angles, namely (δ_{-n}, δ_n). The latter is a consequence of the violation of the ultra-adiabaticity condition (39).

The relativistic adiabatic approximation outlined in the preceding paragraphs corresponds to the following definition of relativistic adiabatic evolution.

Definition 2. A two-component state vector $\Psi^{(q)}(t)$ is said to undergo an *exact adiabatic evolution* if and only if

$$\Psi^{(q)}(t) = \mathrm{e}^{\mathrm{i}\alpha_n(t)}\Psi_n^{(q)}[R(t)] + \mathrm{e}^{\mathrm{i}\alpha_{-n}(t)}\Psi_{-n}^{(q)}[R(t)] \tag{62}$$

for some n and $\alpha_{\pm n}$.

The *relativistic adiabatic approximation* corresponds to the case where (62) is valid approximately, i.e.

$$\Psi^{(q)}(t) \approx \mathrm{e}^{\mathrm{i}\alpha_n(t)}\Psi_n^{(q)}[R(t)] + \mathrm{e}^{\mathrm{i}\alpha_{-n}(t)}\Psi_{-n}^{(q)}[R(t)]. \tag{63}$$

For the cases where the $|\Phi_n\rangle$ are orthogonal, this approximation is valid if and only if $\langle\Phi_m|\dot{\Phi}_n\rangle \approx 0$ for all $m \neq n$.

Definition 2 provides a suitable definition for an adiabatic evolution in relativistic (scalar) quantum mechanics. In particular, it ensures that for a cyclic change of the parameters of the system, the one-component Klein–Gordon field and its time-derivative have cyclic evolutions.

The difference between the ultra-adiabatic and adiabatic evolutions is that for a cyclic ultra-adiabatic evolution the (possibly non-unimodular) 'phases' (complex phase angles) of the one-component field and its time-derivative are required to be equal, whereas in a cyclic adiabatic evolution these phases are generally different. More precisely, if the parameters of the system are periodic, i.e. $R(T) = R(0)$ for some T, the one-component Klein–Gordon field corresponding to (63) satisfies equation (30) with α and β given by

$$\alpha = \gamma_n(T) + \mathrm{i}\left[\xi_n(T) - \ln\left(\frac{\cos[\eta_n(T)/2]}{\cos[\eta_n(0)/2]}\right)\right]$$

$$\beta = \gamma_n(T) + \mathrm{i}\left[\xi_n(T) - \ln\left(\frac{\sin[\eta_n(T)/2]}{\sin[\eta_n(0)/2]}\right)\right]$$

where $\gamma_n(T)$ is the adiabatic geometric phase angle, and η_n and ξ_n are defined by (60) and (61) respectively.

One must also note that since the defining condition for the relativistic and non-relativistic adiabatic evolution are identical, one can use the well known results of non-relativistic quantum mechanics to generalize the above results to the case where E_n is degenerate.

Perhaps the most important aspect of the above derivation of the geometric phase is that it does not use the particular form of an inner product on $\mathcal{H}_t \oplus \mathcal{H}_t$, i.e. the Hermitian matrix h of (10). It only uses the inner product on $\mathcal{H}_t$. A review of the existing literature [10, 21] shows that in the previously studied examples a great deal of effort was made to define an inner product on the space of solutions before the problem of the geometric phase could be addressed. The construction of such an inner product is a highly technical problem and a satisfactory solution for arbitrary (non-stationary) spacetimes is not known. The results of this section indicates that indeed one does not need to construct an inner product on the space of solutions. What is needed is the L^2 inner product on $\mathcal{H}_t$ which is naturally given by the induced spatial metric. In this way, one can conveniently avoid the difficult problem of constructing an inner product on the space of solutions and carry on with the analysis of the adiabatic geometric phase. This is the main practical advantage of the method developed in this article.

6. Rotating magnetic field in a Minkowski background

Consider the geometric phase induced on a Klein–Gordon field in a Minkowski background due to a rotating magnetic field. This problem was originally studied by Anandan and Mazur [10] using the one-component formalism.

For this system, in a global Cartesian coordinate system, one has $g_{00} = -1$, $g_{ij} = \delta_{ij}$, $g_{0i} = V = 0$, and $\mathcal{H}_t = L^2(\mathbb{R}^3)$. Following [10], let us first consider the case of a constant magnetic field along the x^3-axis. Then in the symmetric gauge, one has $A_0 = A_3 = 0$, $A_1 = -Bx^2/2$, and $A_2 = Bx^1/2$. Substituting these equations in equations (3) and (4), one finds $\hat{D}_1 = 0$ and

$$\hat{D}_2 = -\nabla^2 - \mathrm{i}eB\frac{\partial}{\partial\varphi} + \frac{e^2B^2}{4}\rho^2 + \mu^2 \tag{64}$$

where ∇^2 is the Laplacian and (ρ, φ, x^3) are cylindrical coordinates in $\mathbb{R}^3$. Clearly $\hat{D}_2$ is self-adjoint. Therefore, equation (14) reduces to the eigenvalue equation for $\hat{D}_2$, namely Φ_n are orthogonal eigenvectors of $\hat{D}_2$ with eigenvalue E_n^2. Furthermore, if one chooses $q = \mathrm{i}$ in equation (7), then the Hamiltonian $H^{(i)}$ of (5) becomes self-adjoint with respect to the inner product (11).

The situation is quite similar to the non-relativistic Landau level problem. Clearly, Φ_n are infinitely degenerate. They are given by

$$\Phi_n^{(p,m)} = N_n \mathrm{e}^{\mathrm{i}px^3}\mathrm{e}^{\mathrm{i}m\varphi}\chi_{nmp}(\rho) \tag{65}$$

where $p \in \mathbb{R}$, $m = 0, 1, 2, \ldots$, label the vectors within the degeneracy subspace $\mathcal{H}_n$, χ_{nmp} are orthogonal solutions of

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \left(k^2 - \frac{m^2}{\rho^2} - \lambda^2\rho^2\right)\right]\chi_{nmp}(\rho) = 0$$
$$k^2 := E_n^2 - (p^2 + \mu^2 + emB) \qquad \lambda := \frac{eB}{2} \tag{66}$$

and N_n are normalization constants chosen in such a way as to ensure

$$\langle\Phi_{\tilde{n}}^{(\tilde{p},\tilde{m})}|\Phi_n^{(pm)}\rangle = \delta(\tilde{n}, n)\delta(\tilde{m}, m)\delta(\tilde{p}, p). \tag{67}$$

Here $\delta(\ ,\ )$ denotes a Kronecker or a Dirac delta function depending on whether the arguments are discrete or continuous, respectively.

In order to solve the eigenvalue problem for the rotating magnetic field, one can easily use the unitary transformations [10, 35]

$$\mathcal{U}(\theta, \varphi) = \mathrm{e}^{-\mathrm{i}\varphi\hat{J}_3}\mathrm{e}^{-\mathrm{i}\theta\hat{J}_2}\mathrm{e}^{\mathrm{i}\varphi\hat{J}_3} \tag{68}$$

relating the eigenvectors Φ_n of $\hat{D}_2$ to those corresponding to the constant magnetic field (65). In equation (68), θ and φ are polar and azimuthal angles in spherical coordinates and $\hat{J}_i$ are angular momentum operators (generators of $SO(3)$) acting on the Hilbert space $L^2(\mathbb{R}^3)$. $\mathcal{U}(\theta, \varphi)$ are well defined everywhere except along the negative x^3-axis which can be excluded by assuming that $\boldsymbol{B}(t) = (B, \theta(t), \varphi(t))$ does not cross this axis. Otherwise, one may choose another coordinate frame and remedy the problem by performing appropriate gauge transformations as described in [35] for the non-relativistic case. Clearly,

$$\hat{D}_2[\boldsymbol{B}(t)] = \mathcal{U}(\theta(t), \varphi(t))\hat{D}_2[\boldsymbol{B} = B\hat{x}^3]\mathcal{U}^\dagger(\theta(t)\varphi(t))$$
$$\Phi_n[\boldsymbol{B}(t)] = \mathcal{U}(\theta(t), \varphi(t))\Phi_n[\boldsymbol{B} = B\hat{x}^3] \tag{69}$$
$$E_n[\boldsymbol{B}(t)] = E_n[\boldsymbol{B} = B\hat{x}^3] = \text{constant}. \tag{70}$$

The latter relation which implies $\dot{E}_n = 0$ indicates that an adiabatic evolution of this system is, in fact, ultra-adiabatic.

As noted in [10], the presence of the degeneracy leads to non-Abelian geometric phases (24) defined by the connection one-form $\mathcal{A}_n$, (25). The components of $\mathcal{A}_n$ are given by the non-Abelian generalization of (36), namely

$$A_n^{IJ} = \mathrm{i}\langle\Phi_n^{(I)}|\,\mathrm{d}\Phi_n^{(J)}\rangle \tag{71}$$

and are independent of the choice of the matrix h of (10). In equation (71), $I := (p, m)$ and $J := (p', m')$, and use is made of (67). I shall not be elaborating on this problem any further since the specific results are exactly the same as the ones reported in [10]. It is, however, worth mentioning the following.

• Each Φ_n defines a pair of orthonormal two-component eigenvectors $\Psi_{\pm n}^{(i)}$ corresponding to the choices $\pm E_n$ for each eigenvalue E_n^2 of $\hat{D}_2$. Hence in this case the two-component formalism reproduces the results of [10] which were obtained using a more subtle method of taking square root of the second-order Klein–Gordon operator and projecting onto the spaces of negative and positive energy (frequency) solutions of the Klein–Gordon equation.

• For the case where the magnitude of the magnetic field B also changes, the eigenvalues will depend on time, i.e. $\dot{E}_n \neq 0$. This means that in general the ultra-adiabatic and adiabatic approximations have different domains of validity. The former demands both $\dot{E}_n \approx 0$ and $\langle\Phi_m|\dot{\Phi}_n\rangle \approx 0$, for all $m \neq n$, where as the latter only requires the second condition.

7. Rotating cosmic string

In [21], the authors study the geometric (or rather topological) phases induced on a Klein–Gordon field due to a rotating cosmic string. In this section, I shall outline a solution to this problem using the two-component formalism.

The local coordinate expression for the metric corresponding to a rotating cosmic string with angular momentum j and linear mass density d is [21]

$$\mathrm{g} = \begin{pmatrix} -1 & 0 & -4j & 0 \\ 0 & 1 & 0 & 0 \\ -4j & 0 & (\alpha\rho)^2 - (4j)^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{72}$$

where $(x^\mu) = (t, \rho, \varphi, z)$ and (ρ, φ, z) are cylindrical coordinates on the spatial hypersurface Σ_t and $\alpha := 1 - 4d$. Σ_t corresponds to a cone with a deficit angle $\beta = 8\pi d = 2\pi(1-\alpha)$.

Note that for $\rho \leqslant 4j/\alpha$, $\partial/\partial\varphi$ becomes timelike. This leads to the existence of closed timelike curves. This region can be ignored by imposing appropriate boundary conditions on the fields, i.e. $\Phi = 0$ for $\rho \leqslant 4j/\alpha$.

Performing the necessary calculations, one finds the following expressions for the operators $\hat{D}_1$ and $\hat{D}_2$ of equations (3) and (4):

$$\hat{D}_1 = \frac{8j}{(\alpha\rho)^2 - (4j)^2}\frac{\partial}{\partial\varphi} \tag{73}$$

$$\hat{D}_2 = \left(\frac{-1}{1-(\frac{4j}{\alpha\rho})^2}\right)\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{(\alpha\rho)^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} - \mu^2\right]. \tag{74}$$

Therefore the conditions for the self-adjointness of the Hamiltonian $H^{(q)}$ of (5) cannot be met. Let us proceed, however, with considering the eigenvectors of $\Psi_n^{(q)}$ of $H^{(q)}$, (12). For the metric (72), equation (15) takes the form

$$\left\{\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{(\alpha\rho)^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\mathrm{i}8jE_n}{(\alpha\rho)^2}\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial z^2} - \mu^2 + \left[1 - \left(\frac{4j}{\alpha\rho}\right)^2\right]E_n^2\right\}\Phi_n = 0. \tag{75}$$

In view of an observation made in [22] and used in [21], let us write Φ_n in the form $\Phi_n = \exp(\mathrm{i}\zeta\varphi)\phi_n$. Substituting this equation in (75), one finds that for $\zeta = -4jE_n$, ϕ_n satisfies

$$\left\{\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{(\alpha\rho)^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} - \mu^2 + E_n^2\right\}\phi_n = 0. \tag{76}$$

Equation (76) may be obtained from (75) by setting $j = 0$ and replacing Φ_n by ϕ_n. Hence ϕ_n determine the eigenvectors of the Hamiltonian for a non-rotating string of the same mass density. In this case $\hat{D}_1$ vanishes and $\hat{D}_2$ becomes self-adjoint. Therefore, ϕ_n are orthogonal eigenvectors of $\hat{D}_2$, with $j = 0$. If one chooses $q = \mathrm{i}$, then the Hamiltonian becomes self-adjoint with respect to the inner product (11).

In fact, it is not difficult to show that the solutions of equation (76) are of the form

$$\phi_n = N_n \mathrm{e}^{\mathrm{i}pz}\mathrm{e}^{\mathrm{i}m\varphi}J_\nu(k\rho) \tag{77}$$

where N_n are appropriate normalization constants, J_ν are Bessel functions, and

$$k := \sqrt{E_n^2 - (p^2 + \mu^2)} \qquad \nu := m/\alpha.$$

The orthogonality property of ϕ_n carries over to Φ_n with the same energy eigenvalue E_n since the measure of the integration on Σ_t is independent of j. This is because of the identity

$$\det[g] = -\det[^{(3)}g]$$

which holds for any metric with the lapse function $N = 1$, [30]. Note, however, that there are Φ_n with different energy eigenvalues which are not orthogonal.

The situation is analogous to the case of a rotating magnetic field. However, in this case the Klein–Gordon field acquires an Aharonov–Bohm type phase which is topological in nature. As Berry describes in his (by now classic) article [5], the Aharonov–Bohm phase may be viewed as a particular case of a geometric phase. This is done, for the original Aharonov–Bohm system of an electron encircling a confined magnetic flux line, by considering the electron to be localized in a box which is then carried around the flux line. Thus the time-dependence of the system is introduced by choosing a coordinate system centered inside the box. This leads to geometric phases for the energy eigenfunctions. The same result is then applied to the electron wavepacket, simply because the geometric phase is independent of the energy eigenvalues, i.e. all the energy eigenvectors and therefore any linear combinations of them, in particular the one forming the localized electron wavepacket, acquire the same geometric phase which is then shown to be the same as the one discovered by Aharonov and Bohm [36].

[21] uses the analogy between the system of rotating cosmic string and that of Aharonov and Bohm to obtain the corresponding geometric phases. This is, however, not quite justified for arbitrary energy eigenfunctions since as shown below and also in [21], unlike the Aharonov–Bohm system, the induced phase in this case does depend on the energy eigenvalue. Consequently an arbitrary localized Klein–Gordon field which is a superposition of different energy eigenfunctions will not be cyclic. Berry's argument therefore applies only to those 'localized' field configurations which are energy eigenfunctions†.

Next, let us use the analogy of Berry's treatment of the Aharonov–Bohm phase [5] to derive the geometric phase in the framework of the two-component formalism. This is done by changing to a frame centred in a box which circulates around the string at a distance larger than $4j/\alpha$. If R^i are coordinates of the centre of the box and x'^i are the coordinates centred at $R = (R^i)$, then the eigenfunctions are of the form: $\Phi_n(x') = \Phi_n(x - R)$. Substituting this expression in the non-Abelian version of (36), one finds

$$\begin{aligned}\mathcal{A}_n^{IJ} &= \mathrm{i}\langle\Phi_n^I(x-R)\left|\frac{\partial}{\partial R^i}\right|\Phi_n^J(x-R)\rangle\,\mathrm{d}R^i \\ &= \mathrm{i}\int_\Sigma \mathrm{d}\Omega\,\phi_n^{*I}(x-R)\left[-4\mathrm{i}E_n j\phi_n^J(x-R)\,\mathrm{d}R^2 + \frac{\partial}{\partial R^i}\phi_n^J(x-R)\,\mathrm{d}R^i\right] \\ &= 4jE_n\delta_{IJ}\,\mathrm{d}R^2 \end{aligned} \tag{78}$$

where $\mathrm{d}\Omega = \alpha\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z$, I and J stand for possible degeneracy labels corresponding to eigenfunctions, R^2 is the polar angle associated with the centre of the box, and ϕ_n are

† Strictly speaking such a localized field configuration does not exist, for the energy eigenfunctions are solutions of an homogeneous elliptic differential equation. However, the localization in the z-direction is irrelevant for the above discussion, and one may attempt to use the infinite degeneracy arising from the axial symmetry of the problem to construct a wavepacket which is localized only in the ρ- and φ-directions and has definite energy. It is for such a special situation that the analysis of [21] applies. As pointed out by one of the referees, the above construction of localized field configuration may not be free of difficulties related to normalizability of the wavepacket.

assumed to be normalized. For a curve C with winding number N_C, the geometric phase 'angle' is given by

$$\gamma_n = N_C \int_0^{2\pi} \mathcal{A}_n = 8\pi j E_n N_C \tag{79}$$

where the labels I, J and δ_{IJ} have been suppressed for convenience. This is identical with the result of [21]. Note, however, that here I have not been concerned with the consideration of the difficult problem of the choice of an inner product for the space of the solutions of the Klein–Gordon equation ($\mathcal{H}_t \oplus \mathcal{H}_t$), such as the one proposed by Ashtekar and Magnon [37] and apparently 'used' by Corichi and Pierri in [21]. In fact, as I have shown in section 3, the geometric phase is independent of the particular choice of such an inner product. This is also implicit in Corichi and Pierri's derivation of the geometric phase in [21]. Although they discuss the Ashtekar–Magnon scheme in some detail, the final derivation does not use the particular form of the inner product.

It is also worth mentioning that although the eigenvalues E_n may be degenerate, the corresponding geometric phase is still Abelian.

8. Conclusion

In this article I showed that the two-component formalism could be consistently used to investigate the geometric phases associated with charged Klein–Gordon fields. This formalism provides a precise definition of the adiabatic approximation and allows Berry's derivation of the adiabatic geometrical phase to be applied to the relativistic Klein–Gordon fields. In particular, I showed that the computation of the adiabatic geometric phase did not involve the explicit construction of an inner product on the space of the initial conditions, or alternatively the space of solutions of the Klein–Gordon equation. It only required the inner product structure of the Hilbert space $L^2(\Sigma_t)$.

In non-relativistic quantum mechanics, the necessary and sufficient condition for the validity of the adiabatic approximation, $\psi \approx \mathrm{e}^{\mathrm{i}\alpha}|n\rangle$, is $\langle m|\dot{n}\rangle \approx 0$ for $m \neq n$, [31, 4], where $|n\rangle$ are instantaneous eigenvectors of the Hamiltonian. If the Hamiltonian is not self-adjoint then the eigenvectors may not be orthogonal. In this case this condition is generalized to $\langle n|n\rangle\langle m|\dot{n}\rangle - \langle m|n\rangle\langle n|\dot{n}\rangle \approx 0$. A direct generalization of the ansatz $\psi \approx \mathrm{e}^{\mathrm{i}\alpha}|n\rangle$ within the two-component formulation of the Klein–Gordon equation leads to an additional condition on the energy eigenvalues, namely $\frac{\mathrm{d}(qE_n)}{\mathrm{d}t} \approx 0$. If this condition is satisfied then the evolution is said to be ultra-adiabatic. If this condition fails to be fulfilled but the adiabaticity condition (40) is satisfied, then the evolution is said to be adiabatic. The expressions for the geometric phase for the ultra-adiabatic and adiabatic evolutions are identical. The only difference is in the dynamical part of the phase.

I employed the general results of the two-component formulation to study adiabatic geometric phases induced by a rotating magnetic field and a rotating cosmic string. The results were in complete agreement with those of the previous investigations [10, 21], but the analysis was considerably simpler.

Finally, I wish to emphasize that the use of the two-component formulation in the study of the geometric phases associated with scalar fields is more advantageous than the more conventional approaches which are based on a decomposition of the space of solutions into positive and negative frequency subspaces and the construction of a positive definite inner product, e.g. those used in [10, 21]. This has two reasons. First, the conventional methods have apparently missed the fact that one does not need to construct an inner product on the space of Klein–Gordon fields in order to be able to calculate the adiabatic geometric phase.

Hence, a major part of these analyses is concerned with the construction of such an inner product. Secondly, these approaches can only be applied to the stationary spacetimes where such an inner product can be constructed.

An example of a non-stationary spacetime is a spatially homogeneous cosmological background (a Bianchi model). The method developed in this paper can be used to study the cosmologically induced geometric phases. This is done in a companion paper [26].

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